77. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are $BC=a$, $CA=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than $a+b$); from D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on BC extended. Required the area of the ellipse which is isogonal conjugate to the straight line DE with respect to the triangle ABC.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Using trilinear coördinates and letting F be the point where $AD=d$ cuts BC, we get $DF=(d-b\sin C)$, $EF=\sqrt{e^2-(d-b\sin C)^2}=f$.

$\therefore$ The coördinates of D, E are respectively,

$$\{-(d-b\sin C),\ e\cos C,\ e\cos B\} \text{ and}$$
$$\{0,\ -(f-b\cos C)\sin C,\ -(f+c\cos B)\sin B\}.$$

Let $l=e\{(f-b\cos C)\sin C\cos C-(f+c\cos B)\sin B\cos B\}$,

$m=-(d-b\sin C)(f+c\cos B)\sin B$, and

$n=(d-b\sin C)(f-b\cos C)\sin C$.

Then $l\alpha+m\beta+n\gamma=0$, is the equation to DE, and $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, is the equation to the ellipse isogonal conjugate to DE.

Let B be the origin, BC, BA the axes of (x, y).

Then $\alpha=y\sin B$, $\gamma=x\sin B$, $\beta=(ac\sin B-a\alpha-c\gamma)/b$.

$\therefore$ $\beta=\sin B(ac-ay-cx)/b$.

Substituting these values of α, β, γ the equation to the ellipse becomes,

$$clx^2+any^2+(al+cn-bm)xy-aclx-acny=0.$$

Let $\Delta=\dfrac{a^2c^2ln(2-cn-ae)}{4acln-(al+cn-bm)^2}$ be the discriminant of the ellipse.

The two values of z in the equation,

$$z^2+\frac{16(cl+an)\Delta}{\{4acln-(al+cn-bm)^2\}^2}\,z-\frac{64\Delta^2}{\{4acln-(al+cn-bm)^2\}^3}=0,$$

give the values of the squares of the semi-axes.

$\therefore$ Area of ellipse

$$=\frac{8\pi\Delta}{\{4acln-(al+cn-bm)^2\}^{\frac{3}{2}}}=\frac{8\pi a^2c^2ln(2-cn-al)}{\{4acln-(al+cn--bm)^2\}^{\frac{5}{2}}}.$$

II. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.**

The coördinates of D are $(d,\ -\beta_1,\ -\gamma_1)$, and of E, $(0,\ \beta_2,\ -\gamma_2)$, and we then have,

$$e^2=(abc)/(4\Delta^2)\{a(\beta_1+\beta_2)(\gamma_2-\gamma_1)+bd(\gamma_1-\gamma_2)+cd(\beta_1+\beta_2)\} \ldots\ldots\ldots\ldots (1).$$

The equation to the perpendicular to BC through A is

$$\gamma\cos C-\alpha\cos A=0 \ldots\ldots\ldots\ldots (2),$$

and this, passing through D, gives

$$\gamma_1\cos C+d\cos A=0 \ldots\ldots\ldots\ldots (3).$$

We have the constant relation

$$a\alpha+b\beta+c\gamma=2\Delta \ldots\ldots\ldots\ldots (4),$$

and this being satisfied by the coördinates of D and E,

$$ad-b\beta_1-c\gamma_1=2\Delta \ldots\ldots\ldots\ldots (5).$$

$$b\beta_2+c\gamma_2=2\Delta \ldots\ldots\ldots\ldots (6).$$

The equation to DE is

$$\alpha(\beta_1\gamma_2+\beta_2\gamma_1)+\beta d\gamma_2+\gamma d\beta_2=0 \ldots\ldots\ldots\ldots (7),$$

the isogonal conjugate of which is

$$\beta\gamma(\beta_1\gamma_2+\beta_2\gamma_1)+\alpha\gamma d\gamma_2+\alpha\beta d\beta_2=0 \ldots\ldots\ldots\ldots (8),$$

which by the problem is an ellipse.

The area of (8) is expressed by

$$K=2\pi\Delta abc\left\{\begin{vmatrix} 0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2 \\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1) \\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0 \end{vmatrix} \div\right.$$

$$\left.\cdot\begin{vmatrix} 0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2, & -a \\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & -b \\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0, & -c \\ a, & b, & c, & 0 \end{vmatrix}^{\frac{3}{2}}\right\} \ldots (9).$$

β_1, γ_1 are determined by (3) and (5), and then β_2 and γ_2 from (1) and (6), giving K in terms of d and elements of the triangle of reference.

It is not obvious how much of a reduction (9) admits, and I have not attempted any.